

# 7. KNOTS

## §7.1. Knots

To a sailor a knot is something you tie in a rope to hold something together.



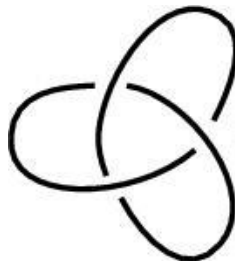
But to a mathematician a knot is a closed path in  $\mathbb{R}^3$  which does not intersect itself.

**Example 1:** The curve:

$$\left. \begin{aligned} x &= 2\cos(2\pi t) - 3\cos(4\pi t) \\ y &= 2\sin(2\pi t) + 3\sin(4\pi t) \\ z &= \sin(6\pi t) \end{aligned} \right\} \text{ for } 0 \leq t \leq 1$$

is a knot. As  $t$  runs from 0 to 1,  $(x, y, z)$  moves along a path from  $(-1, 0, 0)$  in  $\mathbb{R}^3$  and returns to  $(-1, 0, 0)$ . It never passes through the same point until it return to its starting point.

This is one of the two the **trefoil knots**.



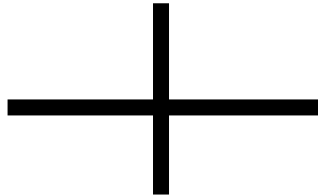
A **knot** is a function  $f:[0, 1] \rightarrow \mathbb{R}^3$  such that:

- (1)  $f$  is continuous;
- (2)  $f(0) = f(1)$ ;
- (3)  $f(s) \neq f(t)$ , for  $s < t$  unless  $s = 0$  and  $t = 1$ .

Condition (1) ensures that there are no breaks in the ‘rope’. Condition (2) ties the two ends together. Condition (3) ensures that different parts of the ‘rope’ don’t pass through each other.

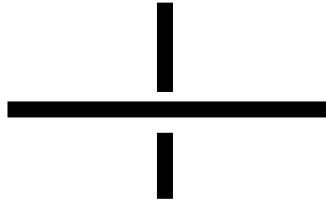
We tie the ends together so that, as we manipulate the knot, it retains its identity. If we had free ends we’d have to keep the ends well apart so that there’d be no danger of the knot coming undone.

It’s difficult to think and work in 3 dimensions so we represent knots by 2-dimensional pictures. However if we take a projection of a knot onto a plane we lose important information. If we’re projecting onto the  $x$ - $y$  plane, and one part of the knot lies directly above or below another, the projection will show the knot intersecting.

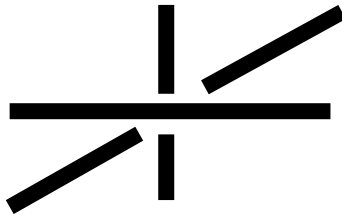


We lose the information as to which part of the knot passes over the top of the other. To remedy this we adopt

the convention that the part that goes underneath is shown as if it is broken.



Of course this convention would fall down if there were three or more parts of the knot which project to the same point on the plane. If one part goes over, another goes in the middle and one part goes underneath the other two, how do we show which is which?



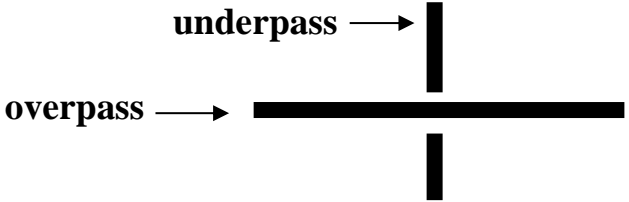
Such a situation (where more than two points project onto the same point) could be called a ‘multiple crossing’. The broken line convention is of use only with ‘single crossings’, that is, where at most two points on the knot project onto the same point.

We could develop a more complicated system to cope with multiple crossings, but that isn’t necessary because the various portions of the knot can be slightly

deformed so that a multiple crossing resolves into several single crossings.

A **knot projection** is the projection of a (three-dimensional) knot onto a plane in such a way that there are no multiple crossings, and with the broken line convention to indicate the part that goes underneath at each crossing.

Since we only deal with single crossings we drop the adjective ‘single’ and simply call them **crossings**. The portion of the knot that passes over the top is called the **overpass** and the other portion is called the **underpass**.



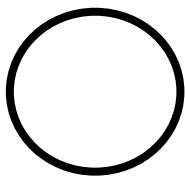
Crossings aren’t an intrinsic feature of knots in 3-dimensions. They only exist when we view a knot from a particular direction, or equivalently, project a knot onto a certain plane. The same knot can have quite different looking projections. Even the number of crossings can be different.

The simplest knot is the function

$$f(t) = (\cos(2\pi t), \sin(2\pi t), 0).$$

Geometrically, this traces out the unit circle

in the  $x$ - $y$  plane. The projection onto the  $x$ - $y$  plane is simply a circle, with no crossings.

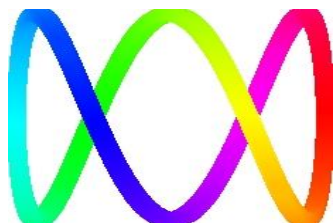


This knot is called the **unknot**. It might seem strange to call something a knot if it isn't knotted, but no more strange than the number zero or the empty set.

Imagine now that we allowed the  $z$  coordinate to vary. We could, for example, take the function

$$g(t) = (\cos(2\pi t), -\sin(6\pi t), \sin(4\pi t)).$$

This has the effect of bending the circle into something resembling a roller-coaster. Looking at it side on we get the ABC logo, a projection with two crossings.



## §7.2. Equivalence of Knots

Two knots are **equivalent** if one can be continuously deformed into the other. More precisely knots are equivalent if and only if they're homotopic as subsets of  $\mathbb{R}^3$ .

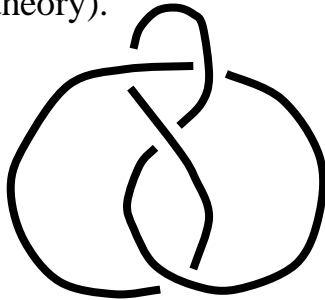
The fundamental question of knot theory is:

**How can you tell whether two knots are equivalent?**

**Example 2:** The following two knots are equivalent.



**Example 3:** The following knot isn't equivalent to the unknot. That is, there's no way we can transform it into a perfect circle (without cutting it open, untying it and joining up the ends again – something which is not permitted in knot theory).

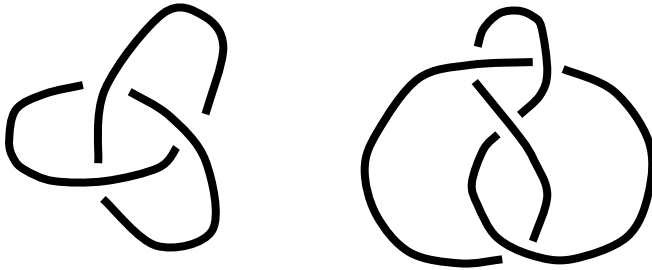


Your intuition would probably convince you that this is so, and you'd be right. But where's the proof? You could make a model of this knot and try to undo it. After a little while you'd become firmly convinced that it can't be done. But that's no proof! What we have to do is to develop some invariants, like we did with surfaces.

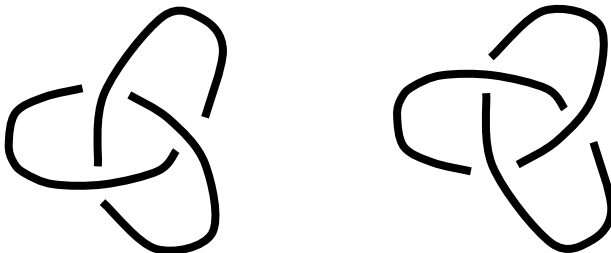
A **knot invariant** is a mathematical object (such as a number, a polynomial or a group) which can be associated with a knot and which remains unchanged when the knot is continuously deformed.

If two knots have different values of a certain invariant they cannot be equivalent. On the other hand if they have the same value that implies nothing.

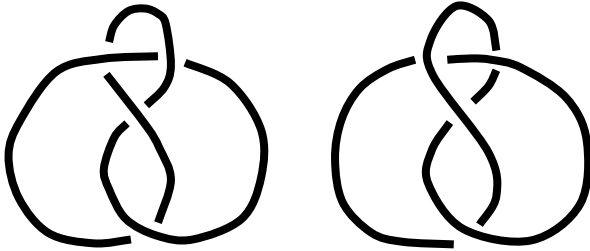
**Example 4:** There is a knot invariant called the Alexander number. The Alexander numbers of the following two knots are 3 and 5 respectively. This will prove that they are not equivalent.



On the other hand the Alexander numbers of the following two knots are both 3. Yet they're still not equivalent. We'd need a much more powerful invariant to distinguish them. These two knots are called the **trefoil knots**.



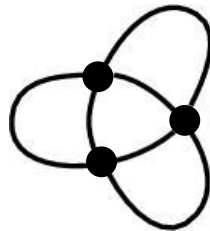
**Example 5:** These knots have four crossings, and very similar looking projections. Yet they are inequivalent.



You can convince yourself of their inequivalence by making rope models, but your inability to transform one into the other is no proof that it can't be done. However the Alexander number *does* distinguish them. The left one has Alexander number 5 while the right one has Alexander number 3 since it is equivalent to one of the two trefoil knots.

### §7.3. Knot Maps

If we ignore the distinction between overpasses and underpasses in a knot projection (and so have no broken lines) we have a map in the plane. Including the outside region we have a map on the sphere. That we call a **knot map**. We'll use the V, E, F notation for the numbers of vertices, edges and faces.



**Example 6:** The trefoil knot map has

$V = 3$  (ie 3 crossings)

$E = 6$

$F = 5$  (including the outside).

Note the distinction between a knot and a knot map. The same knot can have many different knot maps. For example you can scrunch up a knot and introduce additional crossings. So the number of crossings is not an invariant for a knot. The **Crossing Number** of a knot is the minimum number of crossings for any map of the knot.

It's not possible to determine it directly – it can only be found by enumerating knots. If a knot has a map with  $n$  crossings and it is shown to be not equivalent to any knot with fewer crossings then the Crossing Number will be  $n$ . The Crossing Number is used to organise catalogues of knots and the Appendix gives all knots, up to equivalence with Crossing Number up to 7. Also, where a knot is different to its mirror image only one of the two alternatives is shown.

**Theorem 1:** For a knot map with  $n$  crossings the numbers of vertices, edges and faces are given by:

$$V = n$$

$$E = 2n$$

$$F = n + 2.$$

**Proof:** Clearly  $V = n$ .

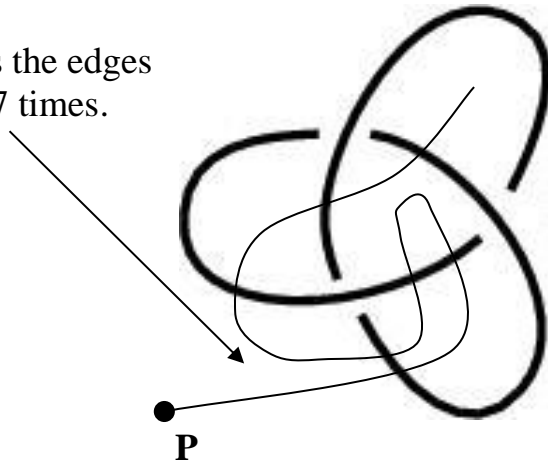
Every vertex has degree 4 so  $E = \frac{4V}{2} = 2n$ .

Finally, by Euler's formula:  $F = E + 2 - V = n + 2$ .

**Theorem 2:** Every knot map can be 2-coloured.

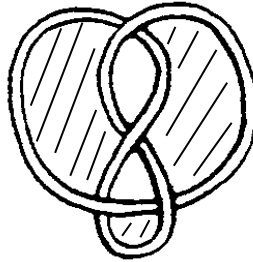
**Proof:** Take a point  $P$  on the plane, but not lying on any edge of the knot. For a given face  $F$  consider the paths that go from  $P$  to some point inside  $F$  (but crossing the edges of the map a finite number of times). If the path goes through a vertex it's deemed to have crossed both edges. For any such path, count the number of times it crosses the edges of the knot.

This path crosses the edges of the knot map 7 times.



As one path is continuously deformed into another this number can change, but only by an even number. So for a given face this number will always be even, irrespective of the path, or it will always be odd. Hence we can colour the map by colouring the faces white if this crossing

number is even and black if it is odd. (The outside of the map is conveniently coloured white.)

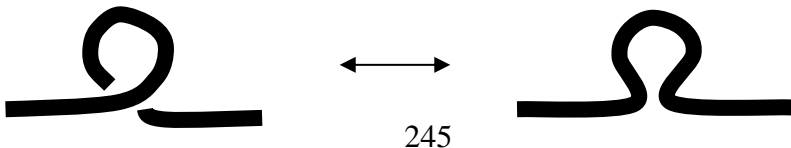


## §7.4. Reidemeister Moves

There's an infinite variety of ways in which one knot can be deformed continuously into another. But there are three basic moves from which all others can be obtained. These are known as the three **Reidemeister moves**. Each of them changes one projection into an equivalent one and each is reversible. The inverse of each Reidemeister move is also a Reidemeister move, of the same type.

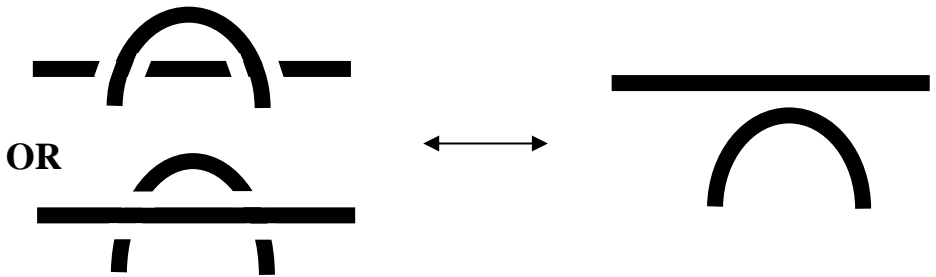
### Reidemeister Move Type I:

If we have a 'kink' in a knot, where the knot returns directly to a crossing without passing through any others, we can remove this crossing, thereby decreasing the number of crossings by 1. Alternatively we can introduce such a 'kink' at any part of the knot and so increase the number of crossings by one.



### Reidemeister Move Type II:

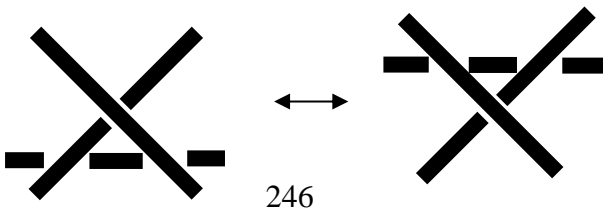
If one portion of the knot, that contains no crossings, passes over (or under) another we can slide it back, thereby reducing the number of crossings by 2. Alternatively we can slide one part of the knot underneath, or over the top of, another and so increase the number of crossings by 2.

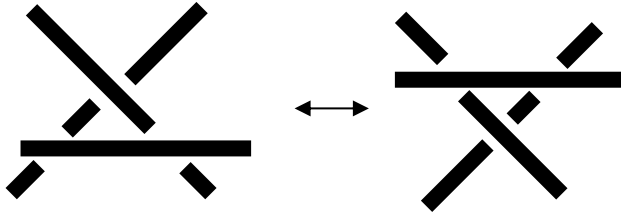


You may wonder why we'd bother with Type I or II Reidemeister moves in the direction that increases the number of crossings. Surely we want to *decrease* the number of crossings to get the knot into as simple a form as possible. That's true. However we may need to increase the number of crossings first in order to move to a projection with fewer crossings than we started with.

### Reidemeister Move Type III:

A type III Reidemeister move doesn't change the number of crossings. Rather it rearranges them.





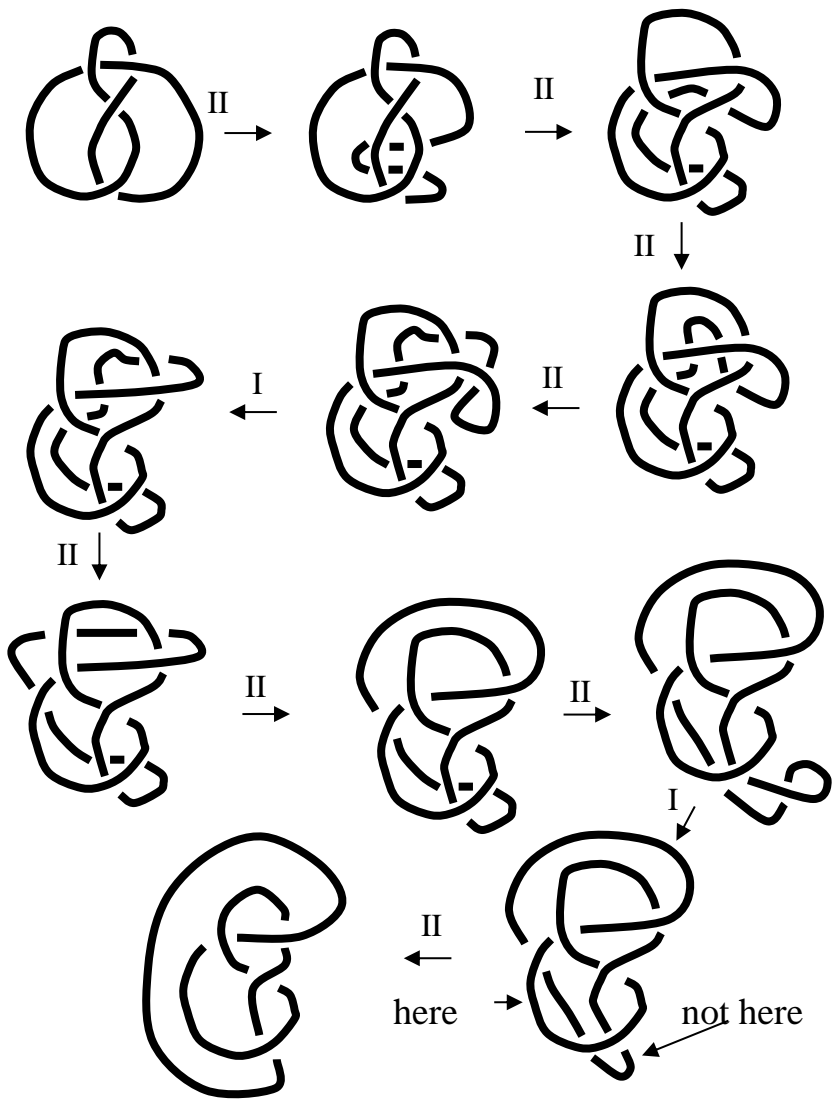
We state the following theorem without proof.

**Theorem 3:** Two knots are equivalent if and only if one can be obtained from the other by a sequence of Reidemeister moves.

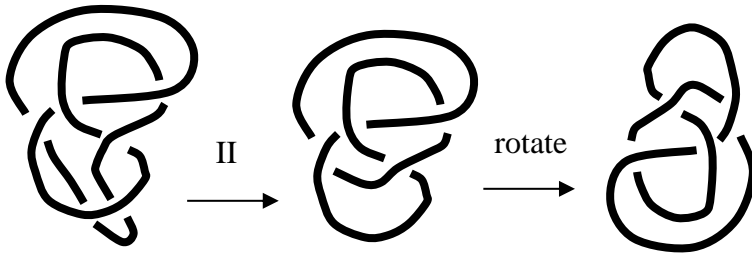
**Example 7:** The right-hand loop of the first knot can be stretched and swung right over the rest of it to produce the second knot.



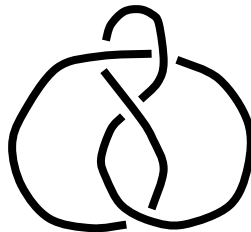
So these knots are equivalent. However the process by which the second knot was obtained isn't one of the three Reidemeister moves. It is, however, equivalent to a *sequence* of Reidemeister moves, as we'll see.



Notice that at the last stage in the above sequence of steps we chose one of two possible Type II Reidemeister moves. Let's see what would have happened if we had chosen to slide the bottom loop up and under and then rotate the whole knot through 180 degrees in the plane of the projection.



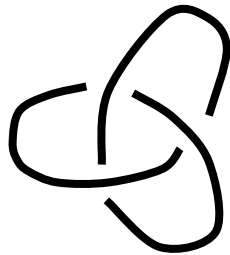
Tidying this up a little we get another figure-eight knot.



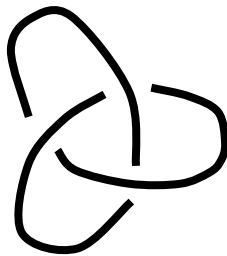
But look carefully. It's not the same figure-eight knot that we started with. It's the mirror image, with under- and over-crossings interchanged. These two versions of the figure-eight knot are called 'conjugates' of one another, and we've shown that they are equivalent. This isn't always the case, though. Many knots are not equivalent to their conjugate.

Now an obvious operation that we can perform on a knot is to turn the knot over, about an axis. Is this a Reidemeister Type 4 move? Not really. The above example shows how a loop can be folded across a knot using Reidemeister moves of types I, II and III. Continuing we can fold the knot right over until it becomes the result of a  $180^\circ$  rotation.

But we must be careful. If we take the trefoil knot



And turn it over about the vertical axis we get:



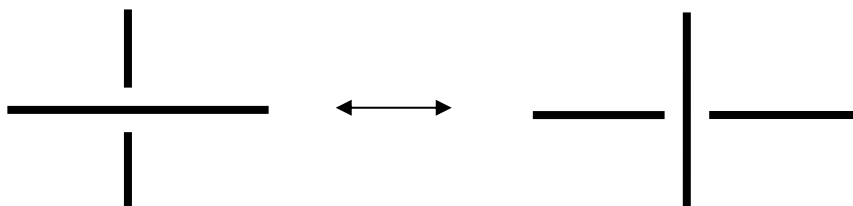
Which is not the same knot. The error lies in treating the knot projection as a 2-dimensional object. It is supposed to represent a 3-dimensional object, and in turning it over, underpasses swap with overpasses. If you take an actual trefoil knot made out of string, and turn it over, you will observe that it is indeed equivalent to the knot we started with.

## §7.5. The Conjugate of a Knot

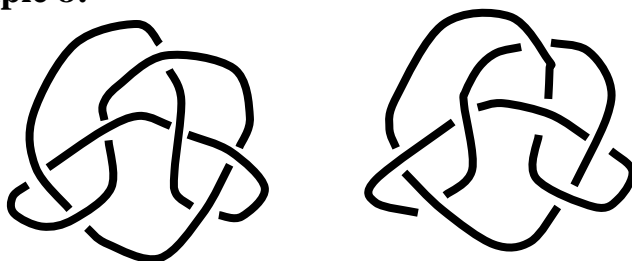
The **conjugate** of a knot  $K$  is its mirror image  $\bar{K}$ .

Note that  $\bar{\bar{K}} = K$ .

A projection of  $\bar{K}$  can be obtained from a projection of  $K$  by changing under-crossings into over-crossings and vice versa.



**Example 8:**

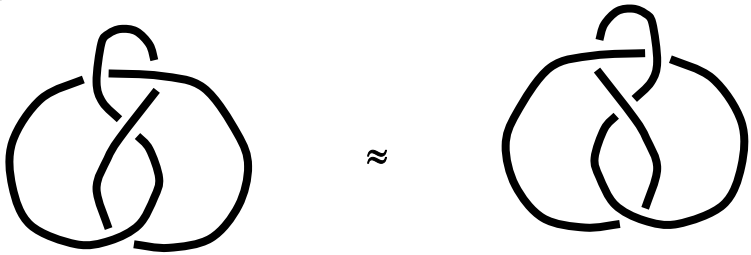


$K$

$\bar{K}$

A knot  $K$  is **amphicheiral** if  $\bar{K} \approx K$ .

**Example 9:** We showed above that the figure 8 knot is amphicheiral.



**Example 10:** The trefoil knots are not amphicheiral. (This is hard to prove and none of the invariants that we'll develop in these notes will be able to distinguish between them.)

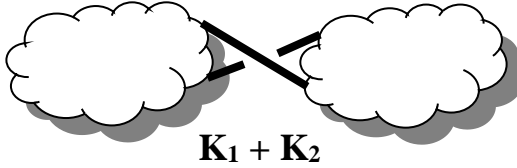


**§7.6. Adding Knots**

To **add** two knots  $K_1$  and  $K_2$  we break each open at some point and join the ends of  $K_1$  with those of  $K_2$  without introducing any additional crossings.



It doesn't matter which end joins which.



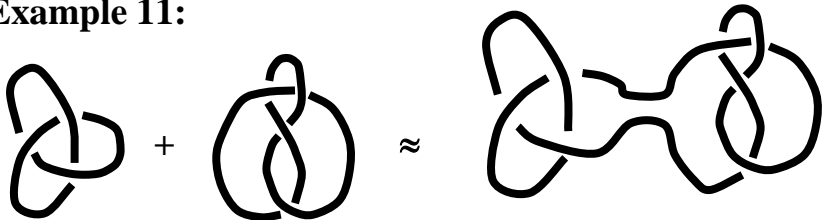
**Theorem 4:** Addition of knots is commutative and associative.

**Theorem 5:**  $O + K \approx K$  for all knots  $K$ . (The unknot is the identity for this addition.)

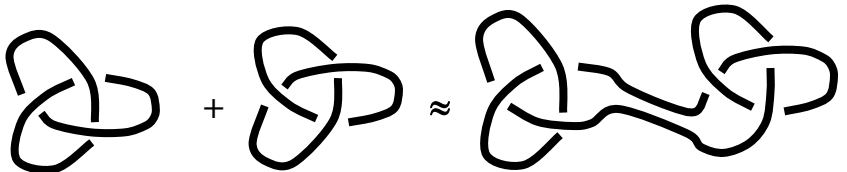


A **prime** knot is one which is not the unknot, and which can't be expressed as a sum  $K_1 + K_2$  where neither is the unknot.

**Example 11:**

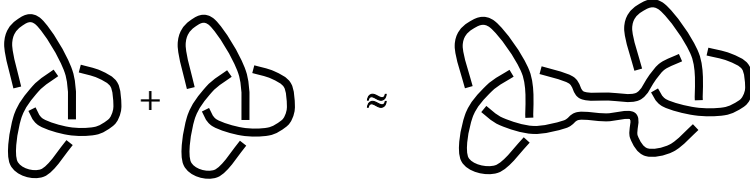


**Example 12:**



The trefoil plus its conjugate is called the **Granny Knot**.

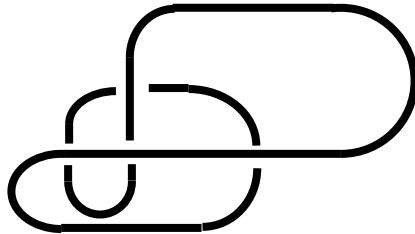
**Example 13:**



The sum of two copies of the same trefoil is called a **Square Knot**.

## EXERCISES FOR CHAPTER 7

**Exercise 1:** Transform the following knot to the unknot using only Type II Reidemeister moves.



**Exercise 2:** The curve:

$$x = 5\cos(2\pi t) - 4\cos(6\pi t);$$

$$y = 4\sin(2\pi t) + 5\sin(6\pi t)$$

for  $0 \leq t \leq 1$  is a closed curve in the  $x$ - $y$  plane. It crosses itself at the following points:

<b>crossing</b>	<b>t</b>		<b>x</b>	<b>y</b>
<b>1</b>	0.0078	0.7597	1.0367	0.9233
<b>2</b>	0.2141	0.7859	3.6224	0
<b>3</b>	0.2403	0.9922	1.0368	- 0.9234
<b>4</b>	0.2597	0.5078	- 1.0367	- 0.9234
<b>5</b>	0.2859	0.7141	- 3.6224	0
<b>6</b>	0.4922	0.7403	- 1.0367	0.9234

Identify the knot

$$x = 5\cos(2\pi t) - 4\cos(6\pi t);$$

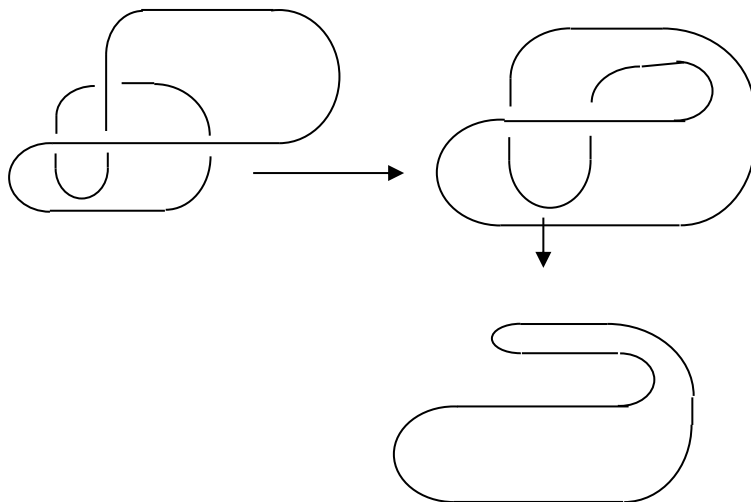
$$y = 4\sin(2\pi t) + 5\sin(6\pi t);$$

$$z = \sin(2\pi r t)$$

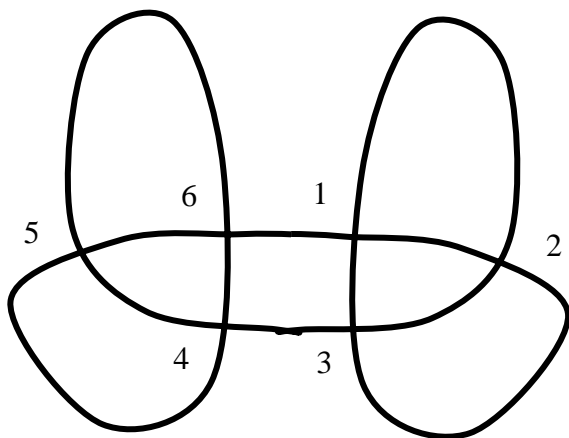
for  $r = 2, 7$  and  $14$ .

# SOLUTIONS FOR CHAPTER 7

## Exercise 1:



## Exercise 2: The plane curve is approximately as follows:

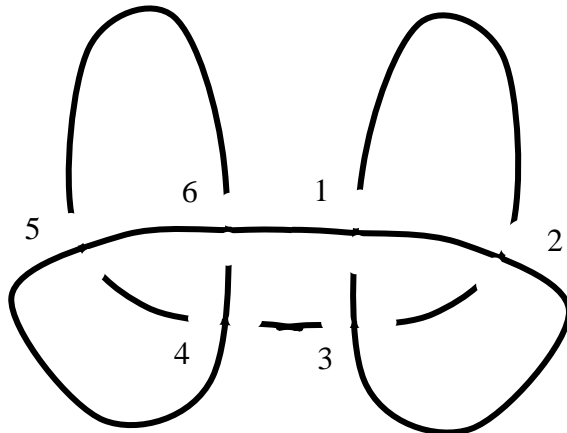


For each value of  $r$  we need to calculate the  $z$ -coordinates of the points at each crossing to determine which lies above which.

X	t	x	y	z		
				r = 2	r = 7	r = 14
1	0.0078	1.0367	0.9233	0.0978	0.3363	0.6335
	0.7597			-1.1215	0.9103	-0.7534
2	0.2141	3.6224	0	0.4359	0.0081	-0.0163
	0.7859			-0.4359	-0.0081	0.0163
3	0.2403	1.0367	-0.9234	0.1215	-0.9103	0.7534
	0.9922			-0.0978	-0.3363	-0.6335
4	0.2597	-1.0367	-0.9233	-0.1215	-0.9103	-0.7534
	0.5078			0.0978	-0.3363	0.6335
5	0.2859	-3.6224	0	-0.4359	0.0081	0.0163
	0.7141			0.4359	-0.0081	-0.0163
6	0.4922	-1.0367	0.9234	-0.0978	0.3363	-0.6335
	0.7405			0.1215	0.9103	0.7534

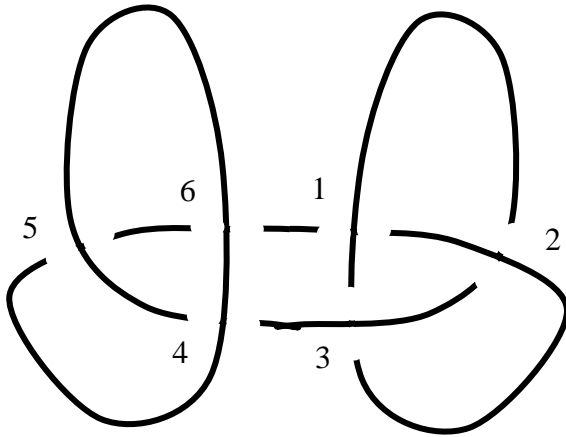
The knots are therefore:

$r = 2$ :



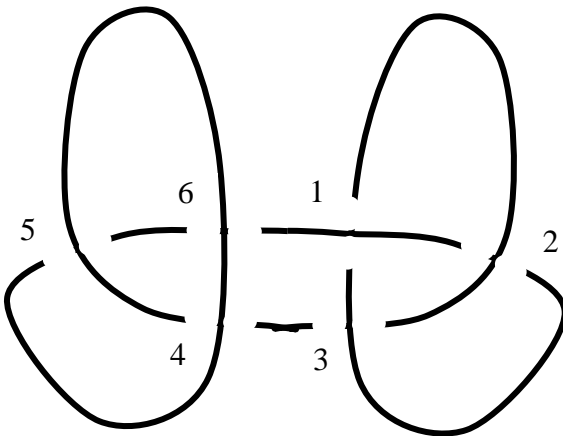
This is the unknot.

$r=7$ :



This is the Granny Knot.

$r = 14$ :



This is the square knot.